SUFFICIENCY CONDITION FOR STABILITY OF A FULLY COUPLED DESIGN PROCESS

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1 INTRODUCTION

The design structure matrix (DSM) is a very strong tool for modelling and analyzing coupled design process (Browning, 2001). For example, the DSM method is applied to an integrated concurrent engineering environment for the conceptual design of a space system with 172 design parameters and 682 dependencies (Avnet and Weigel, 2010). Research on DSMs usually falls in two fields. The first one is the reorganization of DSMs. By diagonalizing or triangularizing a DSM, we can modularize a design process or re-sequence design tasks to shorten the development duration (Eppinger et al., 1994; Kusiak et al., 1994). MacCormack et al. (2006) analyzed the modularity of complex software such as Linux operations system and the Mozilla browser by using the DSM method. The second field is the convergence analysis of coupled design processes based on the DSM model. Smith and Eppinger (1997) did pioneering work by analyzing the convergence of purely parallel design processes based on the DSM model. Mihm et al. (2003) analyzed the oscillation phenomenon in a coupled design process based on a random DSM model and the random matrix theory.

Both in the deterministic and random DSM cases, the dynamics of a coupled design process is captured by a linear difference equation. Namely, the workload of a design task in the current design iteration is the sum of the redesign workloads caused by all the tasks it depends on. Unfortunately, the linear approximation is not so realistic, because it might lead to such a situation that the redesign workload of a task in later design iteration is more than the workload of its original design. As we know, this is impossible. Therefore, in this paper, we construct a new nonlinear dynamic equation for a concurrent fully coupled design process, in which the redesign workload of every design task in later iterations is not more than the workload of its original design. Then we find a sufficiency condition for the asymptotic stability of this coupled design process, and a method of estimating the iteration times that are needed for the workloads of this coupled design process to reduce to an acceptable level. Here, *asymptotic stability* means that design workload goes to zero when time goes to infinity; *fully coupled* means that every design task affects and depends on all the other tasks directly or indirectly. A DSM is not necessarily fully coupled, but once it is diagonalized, the design tasks in its building blocks might be coupled. For example, there exists a coupled block in the DSM of the brake system in Smith and Eppinger (1997). The convergence of these building blocks affects the convergence of the whole design process very much. So the convergence of coupled design process is a very fundamental issue.

2 STABILITY CONDITION

2.1 Model of concurrent fully coupled design process

The dynamics of a concurrent fully coupled design process can be modelled by the following nonlinear difference equations:

$$x_i(n+1) = 1 - \exp\left[-\sum_{j=1}^{I} a_{ij} x_j(n) \right], \quad i = 1, \ldots, I, \tag{1}$$

where I denotes the number of fully coupled design tasks; i the index of design task; n the index of iteration; x_i the design workload of task i, $x_i \in [0, 1]$; a_{ij} the parameter that captures the influence of the workload of task j on the workload of task i. If $a_{ij} > 0$, then task i depends on task j directly; if $a_{ij} = 0$, then task i does not depend on task j directly. Let $\boldsymbol{A} = (a_{ij})_{I\times I}$. Then $\boldsymbol{A}$ is a DSM that captures the concurrent fully coupled design process. From Eq.(1), we can see that the entire design workload of task i caused by the design workloads of other tasks will not be larger than 100%.

2.2 Asymptotic stability of concurrent fully coupled design process

We obtained the sufficiency condition of the asymptotic stability of concurrent fully coupled design process when design iteration times go to infinity.

Theorem 1. The sufficient condition for the asymptotic stability of a concurrent fully coupled design process modelled by the nonlinear difference equations Eq.(1) is that the absolute values of the real parts of all the eigenvalues of the matrix $\boldsymbol{A}^T\boldsymbol{A}$ ($\boldsymbol{A} = (a_{ij})_{I\times I}$) is less than one, i.e.,

$$\left|\mathrm{Re}\left(\lambda_i\left(\boldsymbol{A}^T\boldsymbol{A}\right)\right)\right| < 1,\ i = 1, \ldots, I, \tag{2}$$

Proof. Eq.(1) can be rewritten as

$$\boldsymbol{A}\boldsymbol{x}(n) = -\left\{\ln[1 - x_1(n+1)], \cdots, \ln[1 - x_I(n+1)]\right\}^T, \tag{3}$$

where $\boldsymbol{x}(n) = [x_1(n), \ldots, x_I(n)]^T$. Construct a Lyapunov function of this dynamic system:

$$W(\boldsymbol{x}(n)) = \boldsymbol{x}^T(n)\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}(n)\,. \tag{4}$$

Obviously, $W(\boldsymbol{x}(n))$ is positive definite. Its rate of change is

$$\Delta W(\boldsymbol{x}(n)) = W(\boldsymbol{x}(n)) - W(\boldsymbol{x}(n-1)) = \boldsymbol{x}^T(n)\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}(n) - \sum_{i=1}^{I}\left\{\ln\left[1 - x_i(n)\right]\right\}^2 . \tag{5}$$

According to the Lyapunov stability theorem, if the function $\Delta W(\boldsymbol{x}(n))$ is negative definite, then the nonlinear system Eq.(1) is asymptotically stable at the equilibrium state $(x_1, \ldots, x_I)^T = (0, \ldots, 0)^T$. Let

$$G(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} - \sum_{i=1}^{I}\left\{\ln(1 - x_i)\right\}^2 . \tag{6}$$

Then we have

$$\nabla^2 G(\boldsymbol{x}) = 2\boldsymbol{A}^T\boldsymbol{A} + 2diag\left\{\frac{\ln(1-x_1)-1}{(1-x_1)^2}, \cdots, \frac{\ln(1-x_I)-1}{(1-x_I)^2}\right\} \tag{7}$$

Because $\boldsymbol{A}^T\boldsymbol{A}$ is symmetric, there exists a orthogonal matrix such that $\boldsymbol{U}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{U} = \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_I\}$, where $\lambda_1, \ldots, \lambda_I$ are the eigenvalues of the matrix $\boldsymbol{A}^T\boldsymbol{A}$. So Eq.(7) can be rewritten as

$$\nabla^2 G(\boldsymbol{x}) = 2\boldsymbol{U}diag\left\{\lambda_1 + \frac{\ln(1-x_1)-1}{(1-x_1)^2}, \cdots, \lambda_I + \frac{\ln(1-x_I)-1}{(1-x_I)^2}\right\}\boldsymbol{U}^T . \tag{8}$$

Because $\left[\ln(1-x)-1\right]/(1-x)^2 \le -1$, we know that if $\left|\mathrm{Re}\,\lambda_i\right| < 1$ holds for all i =1, …, I, the matrix $\nabla^2 G(\boldsymbol{x})$ is negative definite. Therefore the function $G(\boldsymbol{x})$ is concave.

If $G(\boldsymbol{x})$ is not negative definite, then there must exist an $\boldsymbol{x}^*$ such that $G(\boldsymbol{x}^*)>0$. So we have

$$G\left(\alpha\boldsymbol{0} + (1-\alpha)\boldsymbol{x}^*\right) > \alpha G(\boldsymbol{0}) + (1-\alpha)G(\boldsymbol{x}^*) = (1-\alpha)G(\boldsymbol{x}^*) > 0 \text{ for } 0 < \alpha < 1. \tag{9}$$

On the other hand, because $\nabla^2 G(\boldsymbol{x})$ is negative definite, we have

$$\begin{aligned} G\left(\alpha\boldsymbol{0} + (1-\alpha)\boldsymbol{x}^*\right) &= G(\alpha\boldsymbol{0}) + (1-\alpha)\nabla G(\boldsymbol{0})\boldsymbol{x}^* + \frac{1}{2}(1-\alpha)^2\boldsymbol{x}^{*T}\nabla G\left(\theta(1-\alpha)\boldsymbol{x}^*\right)\boldsymbol{x}^* \\ &= \frac{1}{2}(1-\alpha)\boldsymbol{x}^{*T}\nabla^2 G\left(\theta(1-\alpha)\boldsymbol{x}^*\right)\boldsymbol{x}^* < 0 \end{aligned} \tag{10}$$

Here arises a contradiction. Therefore, $G(\boldsymbol{x})$ must be negative definite. So the nonlinear system Eq.(1) is asymptotically stable. *Proof Ends.*

Theorem 1 is a sufficient but not a necessary condition of the stability of coupled design process. Namely, if Eq.(2) is satisfied, then the coupled design process must be asymptotically stable. However, it is not necessarily unstable when Eq.(2) is not satisfied. The result of numerical Experiment 1 in Section 3 also shows that. This is because (1) $\left|\mathrm{Re}\,\lambda_i\right| < 1$ is the sufficient but not the necessary condition of the negative definiteness of $G(\boldsymbol{x})$ (or $\Delta W(\boldsymbol{x}(n))$); and (2) even if $G(\boldsymbol{x})$ (or $\Delta W(\boldsymbol{x}(n))$) is not negative definite, the design process might also converge. Therefore, the sufficient condition in Theorem 1 is somewhat too strict. If we use Theorem 1 to evaluate the stability of a coupled design process, we might make an incorrect judgement of considering a stable design process as an unstable one, which can be called *the first type of incorrect judgement*. We need to find a relatively relaxed condition to reduce the possibility of making the first type of incorrect judgement.

We start with Eq.(6), which can be rewritten as

$$G(\boldsymbol{x}) = \sum_{i=1}^{I} \lambda_i \left(\sum_{i=1}^{I} u_{ik} x_k \right)^2 - \sum_{i=1}^{I} \left[\ln(1 - x_i) \right]^2 \tag{11}$$

where u_{ik} is the kth entry of the eigenvector $\boldsymbol{u}_i = (u_{i1}, \ldots, u_{iI})^T$ corresponding to the eigenvalue λ_i. This is because there exists a orthogonal matrix such that $\boldsymbol{U}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{U} = \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_I\}$, where $\boldsymbol{U} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_I)^T$. Actually, the distribution of the $\boldsymbol{x}$ that satisfies $G(\boldsymbol{x})>0$ in the domain [0, 1]×…× [0, 1] determines the stability of the design process. It is not easy to compute this distribution, but we can estimate the distribution in the domain [0, 1]×…× [0, 1] by considering the situation of $x_1 = \ldots = x_I$. In this situation, Eq.(11) becomes

$$G(x_1) = x_1^2 \sum_{i=1}^{I} \lambda_i \left(\sum_{i=1}^{I} u_{ik} \right)^2 - I \left[\ln(1 - x_i) \right]^2 \tag{12}$$

Similar to the analysis in the proof of Theorem 1 (see Eqs.(7)-(10)), we can know that if $\sum_{i=1}^{I} \lambda_i \left(\sum_{i=1}^{I} u_{ik} \right)^2 < I$, then $G(x_1)$ is negative definite, which will give us such a conjecture that the possibility of $G(\boldsymbol{x})>0$ in the domain [0, 1]×…× [0, 1] is very large but there are some $\boldsymbol{x}$ with $G(\boldsymbol{x})>0$ distributed sparsely in this domain. Fortunately, these sparely distributed $\boldsymbol{x}$ with $G(\boldsymbol{x})>0$ have little influence on the stability of the coupled design process. So we have a heuristic rule for evaluating the stability of coupled design process:

Rule 1. If the *judgement parameter* $\sum_{i=1}^{I} \lambda_i \left(\sum_{i=1}^{I} u_{ik} \right)^2 < I$, then the fully coupled design process modelled by the non-linear difference equations Eq.(1) is asymptotically stable.

Rule 1 is neither sufficient nor necessary condition of the stability of coupled design process. However, the results of numerical experiments will show that the possibility of making the first type of incorrect judgement will be reduced by about a half by using Rule 1 compared to using Theorem 1. And the probability of making *the second type of incorrect judgement* (i.e., considering an unstable design process as a stable one) is very close to zero. As we know the probability of making the second type of incorrect judgement when using Theorem 1 is zero, since it is a sufficient condition.

2.3 Stability in finite times of iteration

From the engineering point of view, if the design workloads can reduce to a given level, say, 1%, in finite iterations, then we can consider that the design process is stable. Therefore, we need to find the sufficient condition that the workload of every design task is less than δ after N times of iterations.

Theorem 2. The sufficient condition that the design workload of every design task is less than δ after N times of design iterations is

$$\sum_{j=1}^{I} a_{ij} < \min\left\{-\frac{\ln(1-\delta_n)}{\delta_{n-1}}, n=1,\cdots,N\right\}, \quad i = 1, \ldots, I, \tag{13}$$

where $\delta_0 = 1$, $\delta_N = \delta$ and $\delta < \delta_n < 1$, $n = 1, \ldots, N$.

Proof: Firstly, we try to find the condition that the design workload of every design task is less than δ_1 after one time of design iteration. According to Eq.(1), we let $x_1(0) = \ldots = x_I(0) = 1$ and $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_1)$, $i = 1, \ldots, I$. Then we have $x_i(1) = 1-\exp\left[-\sum_{j=1}^{I} a_{ij}x_j(0)\right] = 1-\exp\left[-\sum_{j=1}^{I} a_{ij}\right]$ $< 1-\exp\left[\ln(1-\delta_1)\right] = \delta_1$, $i = 1, \ldots, I$. So $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_1)$, $i = 1, \ldots, I$, is the sufficient condition for $x_i(1) < \delta_1$, $i = 1, \ldots, I$. Assume that $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_1)$ and $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_n)/\delta_{n-1}$, $n = 2, \ldots, N$-1, $i = 1, \ldots, I$, is the sufficient condition for $x_i(1) < \delta_1$, …, $x_i(N-1) < \delta_{N-1}$, $i = 1, \ldots, I$. Let $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_N)/\delta_{N-1} = -\ln(1-\delta)/\delta_{N-1}$, then we have

$$x_i(N) = 1-\exp\left[-\sum_{j=1}^{I} a_{ij}x_j(N-1)\right] < 1-\exp\left[-\delta_{N-1}\sum_{j=1}^{I} a_{ij}\right] < 1-\exp\left[\ln(1-\delta_N)\right] = \delta_N = \delta \tag{14}$$

for $i = 1, \ldots, I$. Therefore, $\sum_{j=1}^{I} a_{ij} < \min\{-\ln(1-\delta_n)/\delta_{n-1}, n=1,\cdots,N\}$, $i = 1, \ldots, I$, is the sufficient condition for $x_i(1) < \delta_1$, …, $x_i(N-1) < \delta_{N-1}$, $x_i(N) < \delta_N = \delta$, $i = 1, \ldots, I$. *Proof Ends.*

We can also prove the following theorem by contrapositive (the proof is omitted):

Theorem 3. The δ_n, $n = 1, \ldots, N$, which satisfies

$$-\ln(1-\delta_1)/\delta_0 = -\ln(1-\delta_2)/\delta_1 = \cdots = -\ln(1-\delta_N)/\delta_{N-1}, \tag{15}$$

with $1 = \delta_0 > \delta_1 > \cdots > \delta_{N-1} > \delta_N = \delta$, can maximize $\min\{-\ln(1-\delta_n)/\delta_{n-1}, n=1,\cdots,N\}$.

From Theorems 2 and 3, we can get the following Theorem 4 immediately.

Theorem 4. The sufficient condition for the workload of every design task to be less than δ after N times of design iterations is

$$\sum_{j=1}^{I} a_{ij} < -\frac{\ln(1-\delta_1)}{\delta_0} = -\frac{\ln(1-\delta_2)}{\delta_1} = \cdots = -\frac{\ln(1-\delta_N)}{\delta_{N-1}}, \quad i = 1, \ldots, I, \tag{16}$$

where $\delta_0 = 1$, $\delta_N = \delta$ and $\delta < \delta_n < 1$, $n = 1, \ldots, N$.

3 NUMERICAL EXPERIMENTS

3.1 Experiment 1: evaluate the stability by Theorem 1

The first experiment is to explore the correctness of Theorem 1. We generate 100 30-by-30 DSMs randomly, whose entries are a_{ij}s in Eq.(1). Each DSM $\boldsymbol{A}=(a_{ij})_{30\times30}$ is corresponding to a fully coupled design process. We assume that the initial workload of every design task in each design process is 1. Every design process is simulated to see whether the workload of every design task can be less than 0.001 in 1000 times of iterations. If it could, we say that this design process is stable; otherwise, it is unstable. The result of this experiment shows that:

(1) There are 18 DSMs that the maximum real parts of the eigenvalues of the matrix $\boldsymbol{A}^T\boldsymbol{A}$ are larger than one and the corresponding design processes are stable. Namely, if Theorem 1 is applied, then the percentage of making the first type of incorrect judgement is 18%;

(2) There are 0 DSMs that the maximum real parts of the eigenvalues of the matrix $\boldsymbol{A}^T\boldsymbol{A}$ are less than one and the corresponding design processes are unstable. Namely, if Theorem 1 is applied, then the percentage of making the second type of incorrect judgement is 0%.

Generally speaking, the correctness rate is 82% when we judge the stability of fully coupled design process by Theorem 1. The result of the simulation of the 100 design processes is depicted in Figure 1 on Slide 10, in which the '*' or square whose vertical coordinate is 0.5 or 0 means that the design process is stable or unstable respectively. The vertical coordinate of 'o' or 'Δ' represents the maximum real parts of the eigenvalue of $\boldsymbol{A}^T\boldsymbol{A}$. The '*' and the corresponding 'o' means the correct judgements. The square and 'Δ' means the incorrect judgements (the first or the second type).

3.2 Experiment 2: evaluate the stability by Rule 1

The second experiment is to explore the correctness of Rule 1. The DSMs used in this experiment are as the same as in Experiment 1. The result of this experiment shows that:

(1) There are 7 DSMs whose judgement parameter is larger than the dimension the matrix $\boldsymbol{A}^T\boldsymbol{A}$ and whose corresponding design processes are stable. Namely, if Rule 1 is applied, then the percentage of making the first type of incorrect judgement is 7%;

(2) There are 0 DSMs whose judgement parameter is less than the dimension of the matrix $\boldsymbol{A}^T\boldsymbol{A}$ and whose corresponding design processes are unstable. Namely, if Rule 1 is applied, then the percentage of making the second type of incorrect judgement is 0%.

Therefore, the correctness rate is 93% when we judge the stability of fully coupled design process by Rule 1. The result of the simulation of the 100 design processes is depicted in Figure 2 on Slide 12. By comparing the results of Experiments 1 and 2, we know that if Rule 1 is applied, the percentage of making the first type of incorrect judgement is reduced by more than a half compared to Theorem 1.

3.3 Experiment 3: evaluate the stability of the brake system design process

Smith and Eppinger (1997) gave an example of designing a brake system. They construct its DSM model and identify a coupled block (a 28-by-28 matrix, see Figure 3 on Slide 13) in this DSM. Every off-diagonal entry in this coupled block is an estimation of the workload that the design task corresponding to the column of the entry creates for the task corresponding to the row of the entry, which is measured by the percentage of the original design workload. Every entry has three possible values, i.e., 0.5, 0.25 and 0.05, corresponding to strong, medium and weak dependency between design tasks respectively. For this coupled block in the DSM, we apply Rule 1 to evaluate its stability. The value of the judgement parameter is 22.7079, which is less than the dimension of the DSM, i.e., $I = 28$. So the design process should be stable. Simulation based on Eq.(1) supports this judgement.

3.4 Experiment 4: estimate the iteration times

This experiment is to explore the correctness of Theorem 4. We designed an algorithm to solve Eq.(15) to find $\delta_1, \ldots, \delta_{N-1}$ for $N = 2, 3, \ldots, 16$ and $\delta = 0.01$. So the upper bounds of the sum of the entries in every row of the DSM $\boldsymbol{A}$ are 0.102835, 0.225615, 0.334969, 0.42467, 0.497288, 0.556402, 0.604986, 0.645380, 0.67947, 0.708431, 0.733246, 0.754875, 0.773712, 0.790241 and 0.804921 for $N = 2, \ldots, 16$ respectively. The relation between the upper bound of the sum of the entries in DSM rows and the maximum iteration times that are needed for the workloads of design tasks to become less than 0.01, which are estimated by using Theorem 4, is depicted by the stepwise curve in Figure 4 on Slide 15. The "*" in Figure 4 represents the actual iteration times that are needed for the workloads of design tasks to become less than 0.01, which are obtained by simulation. From Figure 4, we can see that actual iteration times are always less than or equal to the estimated iteration times, which supports the correctness of Theorem 4. We can also see that the difference between the actual and estimated iteration times is small if the upper bound of the sum of the entries in DSM rows is small. However, the difference becomes large if this upper bound becomes large. This is because Theorem 4 is only a sufficient condition but not a necessary condition.

4 CONCLUSION

In this paper, the stability of concurrent fully coupled design process is investigated. We model a fully coupled design process by a group of nonlinear difference equations. Based on this model, two conclusions of judging its stability are drawn. The first one is the sufficient condition of the asymptotic stability of a fully coupled design process, which is that the maximum real part of the eigenvalue of the matrix $\boldsymbol{A}^T\boldsymbol{A}$ must be less than 1. To reduce the possibility of making the first type of incorrect judgement, a heuristic rule is also proposed. The second one is the sufficient condition that the workload of every design task reduces to a given level after a given times of iteration. Numerical experiments show that the two conclusions are correct. However, we only got the sufficient conditions. Incorrect judgement rate by using Theorem 1 or Rule 1 still exists. And we usually overestimate the iteration times that are needed for design workload to become less than a given level by using Theorem 4 when the maximum value of the sum of the entries in every row of the DSM is large. Therefore, the work we are going to do in the next step is to find the necessary condition of evaluating the stability of coupled design processes, and the method of more accurately estimating the iteration times that is needed for the workload of every design task to reduce to a given level, which will be helpful to planning product developing projects.

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REFERENCES

Avnet, M. S., & Weigel, A. L. (2010). An application of the design structure matrix to integrated concurrent engineering. *Acta Astronautica*, *66*, 937-949.

Browning, T. R. (2001). Applying the design structure matrix to system decomposition and integration problems: a review and new directions. *IEEE Transaction on Engineering Management*, *48*(3), 292-306.

Eppinger, S. D., Whitney, D. E., Smith, R. P., & Gebala, D. A. (1994). A model-based method for organizing tasks in product development. *Research in Engineering Design*, *6*(1), 1-13.

Kusiak, A., Larson, N, & Wang, J. (1994). Reengineering of design and manufacturing processes. *Computers and Industrial Engineering*, *26*(3), 521-536.

MacCormack, A., Rusank, J, & Baldwin, C. Y. (2006). Exploring the structure of complex software design: an empirical study of open source and proprietary code. *Management Science*, *52*(7) 1015-1030.

Mihm, J., Loch, C., & Huchzermeier, A. (2003). Problem-solving oscillations in complex engineering projects. *Management Science*, *49*(6), 733-750.

Smith, R. P., & Eppinger, S. D. (1997). Identifying controlling features of engineering design iteration. *Management Science*, *43*(3), 276-293.

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Sufficiency Condition for Stability of a Fully Coupled Design Process

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Introduction

- Research on Design Structure Matrix (DSM)
  - Reorganization: diagonalization (Eppinger et al., 1994) and triangularization (Kusiak et al. 1994)
  - Convergence analysis: deterministic DSM (Smith and Eppinger, 1997) and random DSM (Mihm et al., 2003)
- Existing model of the dynamics of coupled design process
  - Linear difference equations:

$$x_i(n+1) = \sum_{j=1}^{I} a_{ij} x_j(n) \,, \qquad i = 1, \ldots, I.$$

- New problems
  - How to construct a new model for coupled design process so that the redesign workload of a task in later design iteration is not more than the workload of its original design?
  - What is the stability condition of a coupled design process based on the new model?
  - How to estimate the iteration times that are needed for the workloads of the coupled design process to reduce to an acceptable level?

A nonlinear model of fully coupled design process

$$x_i(n+1) = 1 - \exp\left[-\sum_{j=1}^{I} a_{ij} x_j(n) \right] \qquad i = 1, \ldots, I.$$

- I : number of design tasks;
- i : index of design task;
- n : index of design iteration;
- x_i : design workload of design task i, $x_i \in [0,1]$, $i = 1, \ldots, I$.
- a_{ij} : the parameter that captures the influence of the workload of task j on the workload of task i. If $a_{ij} > 0$, then then task i depends on task j directly; if $a_{ij} = 0$, then task i does not depend on task j directly.
- $A=(a_{ij})_{I\times I}$: the DSM.

Asymptotic stability of concurrent fully coupled design process

- **Theorem 1.** The sufficient condition for the asymptotic stability of a concurrent fully coupled design process modelled by the nonlinear difference equations Eq.(1) is that the absolute values of the real parts of all the eigenvalues of the matrix $A^T A$ is less than one, i.e.,

$$\left|\mathrm{Re}\left(\lambda_i\left(\boldsymbol{A}^T\boldsymbol{A}\right)\right)\right| < 1\ , \qquad i = 1, \ldots, I.$$

- ***Main idea of the proof.*** Use the Lyapunov stability theory.

The Lyapunov function: $W(\boldsymbol{x}(n)) = \boldsymbol{x}^T(n)\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}(n)$

$$\Delta W(\boldsymbol{x}(n)) = W(\boldsymbol{x}(n)) - W(\boldsymbol{x}(n-1)) = \boldsymbol{x}^T(n)\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}(n) - \sum_{i=1}^{I}\left\{\ln\left[1-x_i(n)\right]\right\}^2 \rightarrow$$

$$\boldsymbol{U}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{U} = \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_I\}$$

$$G(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} - \sum_{i=1}^{I}\left\{\ln(1-x_i)\right\}^2 \rightarrow$$

$$\nabla^2 G(\boldsymbol{x}) = 2\boldsymbol{U}diag\left\{\lambda_1 + \frac{\ln(1-x_1)-1}{(1-x_1)^2}, \cdots, \lambda_I + \frac{\ln(1-x_I)-1}{(1-x_I)^2}\right\}\boldsymbol{U}^T$$

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Asymptotic stability of concurrent fully coupled design process

- **Rule 1.** If the judgement parameter

$$\sum_{i=1}^{I}\lambda_i\left(\sum_{i=1}^{I}u_{ik}\right)^2 < I$$

then the fully coupled design process modelled by the non-linear difference equations Eq.(1) is asymptotically stable.

- ***Main idea of the derivation.*** Analyze the distribution of the $\boldsymbol{x}$ that satisfies $G(\boldsymbol{x})>0$ in the domain $[0, 1]\times\ldots\times[0, 1]$.

$$G(\boldsymbol{x}) = \sum_{i=1}^{I}\lambda_i\left(\sum_{i=1}^{I}u_{ik}x_k\right)^2 - \sum_{i=1}^{I}\left[\ln(1-x_i)\right]^2$$

λ_i: the eigenvalue; $\boldsymbol{u}_i = (u_{i1}, \ldots, u_{iI})^T$: the eigenvector; $\boldsymbol{U} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_I)^T$.

Consider the situation of $x_1 = \ldots = x_I$:

$$G(x_1) = x_1^2\sum_{i=1}^{I}\lambda_i\left(\sum_{i=1}^{I}u_{ik}\right)^2 - I\left[\ln(1-x_i)\right]^2$$

Stability in finite times of iteration

- **Theorem 2.** The sufficient condition that the design workload of every design task is less than δ after N times of design iterations is

$$\sum_{j=1}^{I} a_{ij} < \min\left\{ -\frac{\ln(1-\delta_n)}{\delta_{n-1}}, n = 1, \cdots, N \right\}$$

where $\delta_0 = 1$, $\delta_N = \delta$, and $\delta < \delta_n < 1$, $n = 1, \ldots, N$.

- ***Main idea of the proof.*** Assume that $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_1)$ and $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_n)/\delta_{n-1}$, $n = 1, \ldots, N$, $i = 1, \ldots, I$, is the sufficient condition for $x_i(1) < \delta_1, \ldots, x_i(N-1) < \delta_{N-1}$, $i = 1, \ldots, I$. Let $\sum_{j=1}^{I} a_{ij} < -\ln(1-\delta_N)/\delta_{N-1} = -\ln(1-\delta)/\delta_{N-1}$

Then we have

$$x_i(N) = 1 - \exp\left[-\sum_{j=1}^{I} a_{ij} x_j(N-1) \right] < 1 - \exp\left[-\delta_{N-1} \sum_{j=1}^{I} a_{ij} \right] < 1 - \exp\left[\ln(1-\delta_N) \right] = \delta_N = \delta$$

for $i = 1, \ldots, I$.

Stability in finite times of iteration

- **Theorem 3.** The δ_n, $n = 1, \ldots, N$, which satisfies

$$-\ln(1-\delta_1)/\delta_0 = -\ln(1-\delta_2)/\delta_1 = \cdots = -\ln(1-\delta_N)/\delta_{N-1}$$

with $1 = \delta_0 > \delta_1 > \ldots > \delta_{N-1} > \delta_N = \delta$, can maximize

$$\min\{-\ln(1-\delta_n)/\delta_{n-1}, n = 1, \cdots, N\}$$

- **Theorem 4.** The sufficient condition for the workload of every design task to be less than δ after N times of design iterations is

$$\sum_{j=1}^{I} a_{ij} < -\frac{\ln(1-\delta_1)}{\delta_0} = -\frac{\ln(1-\delta_2)}{\delta_1} = \cdots = -\frac{\ln(1-\delta_N)}{\delta_{N-1}}, \quad i = 1, \ldots, I,$$

where $\delta_0 = 1$, $\delta_N = \delta$, and $\delta < \delta_n < 1$, $n = 1, \ldots, N$.

Numerical experiments

- **Experiment 1: evaluate the stability by Theorem 1**
  - Generate 100 30×30 DSMs (*A*) randomly
  - There are 51 DSMs that the maximum real parts of the eigenvalues of the matrix A^TA are less than one, and 49 DSMs that the maximum real parts of the eigenvalues of the matrix A^TA are larger than one.
  - Minimum eigen value: 0.8917; maximum eigen value: 1.1000.
  - Result:
    - The first type of incorrect judgement: 18%;

      (Max $\left|\mathrm{Re}\left(\lambda_i\left(A^T A\right)\right)\right| \geq 1$ but stable).
    - The second type of incorrect judgement: 0%.

      (Max $\left|\mathrm{Re}\left(\lambda_i\left(A^T A\right)\right)\right| < 1$ but unstable).

MANAGING COMPLEXITY
BY MODELLING DEPENDENCIES

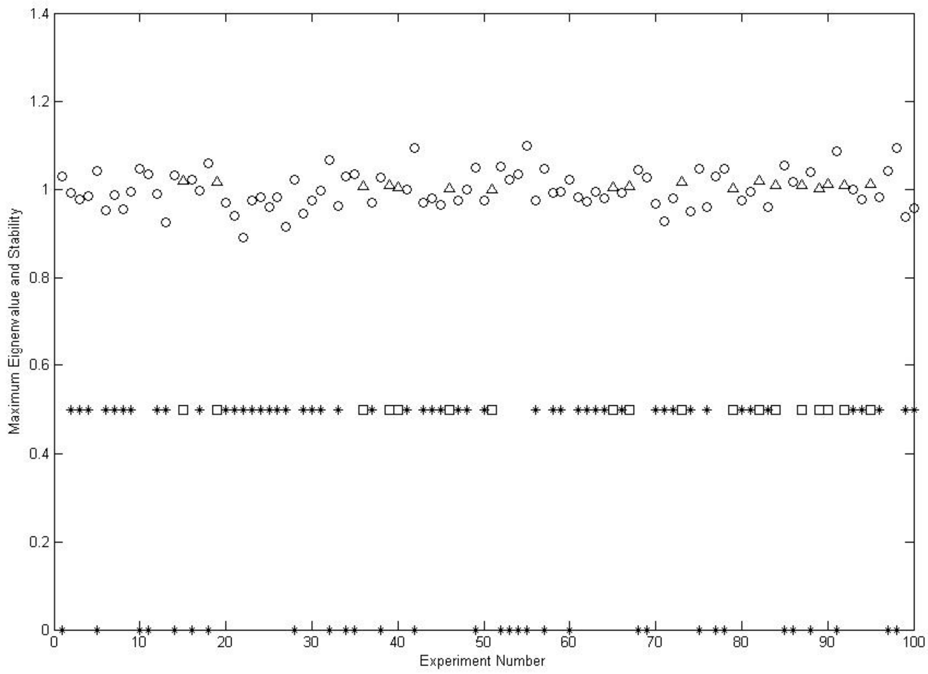

Figure 1. Stability judgment of fully coupled design process by using Theorem 1.

Numerical experiments

- **Experiment 2: evaluate the stability by Rule 1**
  - Generate 100 30×30 DSMs (*A*) randomly
  - There are 56 DSMs whose judgement parameters are less than I, and 44 DSMs whose judgement parameters are larger than one.
  - Minimum judgement parameter: 27.0148;
    Maximum judgement parameter: 32.2306.
  - Result:
    - The first type of incorrect judgement: 7%,
      i.e., $\sum_{i=1}^{I}\lambda_i\left(\sum_{i=1}^{I}u_{ik}\right)^2 \geq I$ but stable.
    - The second type of incorrect judgement: 0%,
      i.e., $\sum_{i=1}^{I}\lambda_i\left(\sum_{i=1}^{I}u_{ik}\right)^2 < I$ but unstable.

MANAGING COMPLEXITY
BY MODELLING DEPENDENCIES

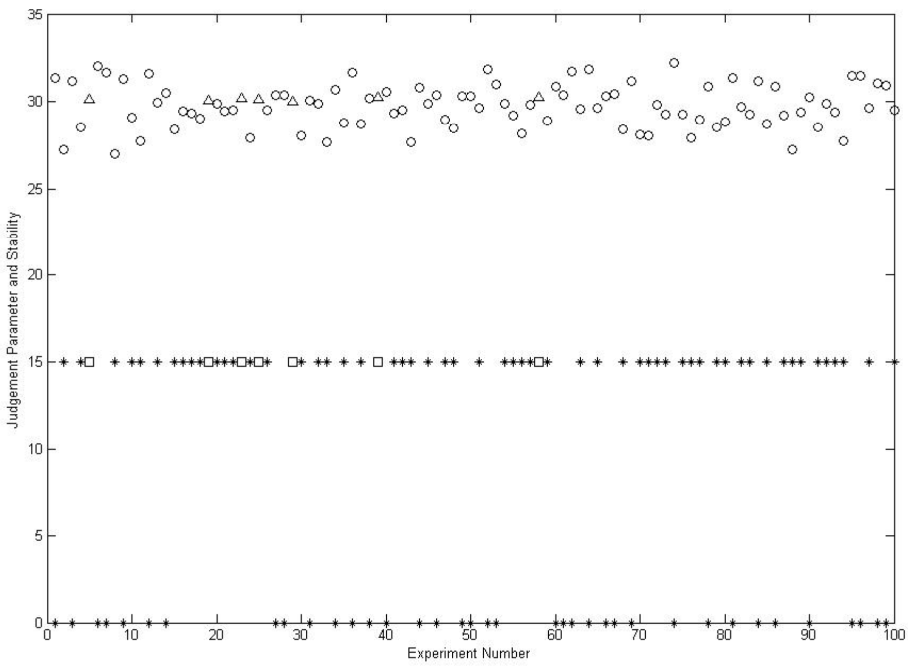

Figure 2. Stability judgment of fully coupled design process by using Rule 1.

Numerical experiments

- **Experiment 3: evaluate the stability of brake system design process**

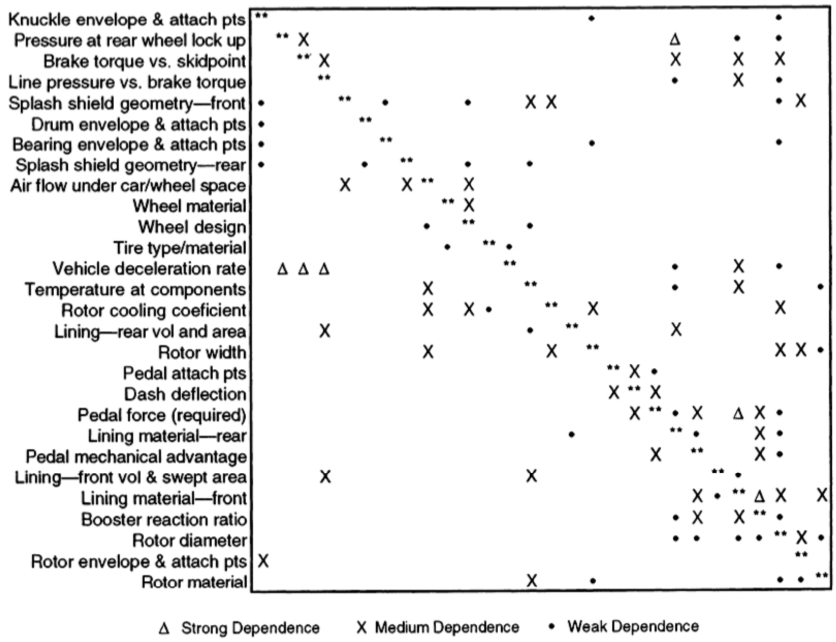

Figure 3. DSM of the brake system design process (Smith and Eppinger, 1997). Judgement parameter =22.7079 < l = 28

Δ : 0.5
× : 0.25
● : 0.05

Numerical experiments

- **Experiment 4: estimate the iteration times**
  - Solve Eq.(15) to find $\delta_1, \ldots, \delta_{N-1}$ for $N = 2, 3, \ldots, 16$ and $\delta = 0.01$.
    N: expected iteration time;
    δ: the final workload of every design tasks.
  - If the upper bounds of the sum of the entries in every row of the DSM is less than:
    0.102835, 0.225615, 0.334969, 0.424670, 0.497288,
    0.556402, 0.604986, 0.645380, 0.679470, 0.708431,
    0.733246, 0.754875, 0.773712, 0.790241, 0.804921,
    then the estimated iteration times are $N = 2, \ldots, 16$ respectively.
  - Results:
    - Actual iteration times are always less than or equal to the estimated iteration times;
    - The difference between the actual and estimated iteration times is small if the upper bound of the sum of the entries in DSM rows is small.
    - The difference becomes large if this upper bound becomes large.

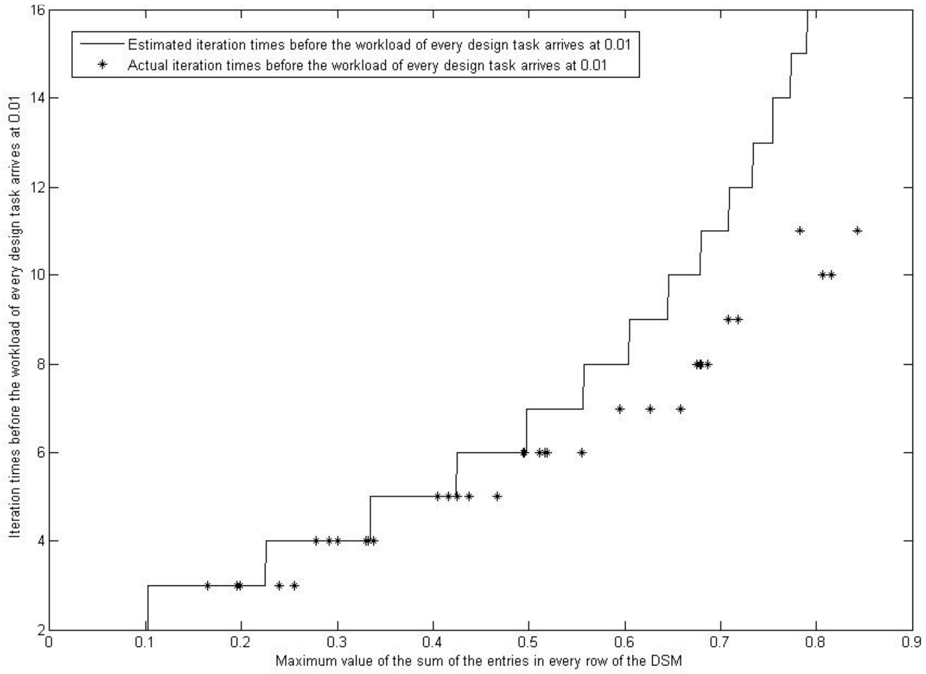

Figure 4. Estimation of iteration times by using Theorem 4.

Conclusion

- Results
  - A sufficient condition of the asymptotic stability of a fully coupled design process;
  - A heuristic rule to reduce the possibility of making the first type of incorrect judgement;
  - A sufficient condition for the workload of every design task to reduce to a given level after a given times of iteration.
- Open problems:
  - The necessary condition of the asymptotic stability of a fully coupled design process;
  - The necessary condition for the workload of every design task reduces to a given level after a given times of iteration.

Acknowledgement

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